

On the symmetry of commuting differential operators with singularities along hyperplanes

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Abstract

We study the commutants of a Schrödinger operator whose potential function possesses inverse square singularities along some hyperplanes passing through the origin. It is shown that the Weyl group symmetry of the potential function and the commutants naturally results from such singularities and the generic nature of the coupling constants.

1 Introduction

The Calogero-Moser-Sutherland models and their generalizations developed by Olshanetsky and Perelomov are completely integrable systems with long-range interactions. These systems are closely related to root systems and Weyl groups. Let (Σ, W) be a pair consisting of a root system and the corresponding Weyl group. In the quantum case, the Schrödinger operator is

$$-\Delta + \sum_{\alpha \in \Sigma^+} m_\alpha(m_\alpha + 1) \langle \alpha, \alpha \rangle u(\langle \alpha, x \rangle), \quad (1.1)$$

where $\Delta = \sum_{i=1}^n \partial^2 / \partial x_i^2$, $\langle u, v \rangle$ is the standard inner product of $u, v \in \mathbf{R}^n$, the quantities m_α are W -invariant parameters, and $u(t) = t^{-2}$ (rational case), $\omega^2 \sinh^{-2} \omega t$, $\omega^2 \sin^{-2} \omega t$ (trigonometric cases) or $\wp(t)$ (elliptic case). Obviously, this operator is W -invariant. In addition to this operator, there are well-known conserved operators for such a system that are W -invariant. In the rational potential case, we can consider integrable systems invariant under the action of finite Coxeter groups.

It is evident that the potential function of the above Schrödinger operator possesses inverse-square singularities along the reflection hyperplanes of W . The main object of this paper is to show that the Weyl group (or Coxeter group) symmetry of such a system results naturally from the inverse square singularities and the generic nature of the parameters m_α .

To make the following discussion more precise, we introduce some notation. For a non-zero vector $\alpha \in \mathbf{R}^n$, we denote by H_α the hyperplane $\langle \alpha, x \rangle = 0$ and by r_α reflection with respect to H_α . For a finite set \mathcal{H} of mutually non-parallel vectors in \mathbf{R}^n , let L be the Schrödinger operator defined by

$$L = -\Delta + R(x), \quad R(x) = \sum_{\alpha \in \mathcal{H}} \frac{C_\alpha}{\langle \alpha, x \rangle^2} + \tilde{R}(x), \quad (1.2)$$

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where $\tilde{R}(x)$ is real analytic at $x = 0$ and the constants C_α are non-zero for $\alpha \in \mathcal{H}$. We call \mathcal{H} the *hyperplane arrangement* of L or the *hyperplane arrangement* of $R(x)$.

Assume that there exists a commutant P of L with constant principal symbol. Note that we do *not* assume the symmetry of either $R(x)$ or P , nor do we assume \mathcal{H} to be a subset of a root system. The first result is stated as follows.

Theorem 1.1 *If $C_\alpha \neq k(k+1)\langle\alpha, \alpha\rangle$ for any integer k , then the principal symbol of P is r_α -invariant.*

We prove this theorem in Sections 2 and 3.

We call the potential function $R(x)$ *generic* if $C_\alpha \neq k(k+1)\langle\alpha, \alpha\rangle$ for any integer k and for any $\alpha \in \mathcal{H}$. In non-generic cases, many interesting phenomena have been observed. For example, if the parameters m_α are integers, then there exist W -non-invariant conserved operators for (1.1), in addition to the W -invariant ones [5, 13]. Also in non-generic cases, Veselov, Feigin and Chalykh found new completely integrable systems like (1.1), but whose hyperplane arrangements are not root systems but deformed root systems [4, 12].

Though non-generic cases like those mentioned above are very interesting, we restrict our attention to generic cases beginning in Section 4. In Section 4, we address the problem of determining the permissible kinds of hyperplanes arrangements. To avoid unnecessary complication, we assume the “irreducibility” of \mathcal{H} (Definition 4.1). The main result in Section 4 is that if \mathcal{H} is irreducible, the potential function is generic, and L has a non-trivial commutant, then \mathcal{H} must be a subset of the positive root system of some finite reflection group (Theorem 4.4).

In Sections 5, 6 and 7, we determine the potential function $R(x)$ in the case that the root system containing \mathcal{H} is of the classical type. The type A case is treated in Section 6, and the types B and D are treated in Section 7. We now give a brief summary of the arguments given in those sections. In the case that the root system Σ containing \mathcal{H} is of type A , B or D , we assume that the Schrödinger operator (1.2) commutes with a differential operator P , whose principal symbol is $\sum_{i < j < k} \xi_i \xi_j \xi_k$ for type A and $\sum_{i < j} \xi_i^2 \xi_j^2$ for types B and D . Under this assumption, we can show that \mathcal{H} must coincide with Σ^+ and that the potential function $R(x)$ must be Weyl group invariant.

In [7], [8] and [10], Ochiai, Oshima and Sekiguchi classified the potential functions $R(x)$ satisfying the relation $[-\Delta + R(x), P] = 0$, which do not necessarily possess poles along hyperplanes, in the Weyl group invariant context. They also proved that $-\Delta + R(x)$ is completely integrable for such $R(x)$. In Sections 6 and 7, it is shown that our potential function $R(x)$ and the commutant P are identical to those that they considered. Therefore, it is seen that $R(x)$ is one of the functions classified in [8] (Theorem 6.2, Remark 7.8), and L is completely integrable. Hence, the complete integrability of L essentially follows from the generic nature of coupling constants and the existence of one non-trivial commutant P .

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2 Rank-one reduction

To begin, we introduce some notation. Let $\{e_1, \dots, e_n\}$ be the standard basis of \mathbf{R}^n and $x = (x_1, \dots, x_n)$ be the corresponding coordinates. For simplicity, denote by ∂_i the partial differential $\partial/\partial x_i$ and define $\partial = (\partial_1, \dots, \partial_n)$. We denote the norm of a vector $v \in \mathbf{R}^n$ by $|v|$. An m_0 -th order differential operator P is expressed as

$$P = \sum_{k=0}^{m_0} P_k, \quad P_k = \sum_{|p|=m_0-k} a_p(x) \partial^p,$$

where $p = (p_1, \dots, p_n) \in \mathbf{N}^n$ is a multi-index, and $|p|$ is the length $\sum_i p_i$ of p . Corresponding to this operator, we introduce

$$\tilde{P}_k = \sum_{|p|=m_0-k} a_p(x) \xi^p \quad \text{and} \quad \tilde{P} = \sum_{k=0}^{m_0} \tilde{P}_k \quad (\xi = (\xi_1, \dots, \xi_n)),$$

and call them the *symbols* of P_k and P , respectively. In particular, \tilde{P}_0 is called the *principal symbol* of P .

In this section, we reduce the proof of Theorem 1.1 to that for the rank-one rational case. For this purpose, we introduce a new coordinate system on \mathbf{R}^n . First, choose $\alpha \in \mathcal{H}$. Then, let $e'_1 = |\alpha|^{-1} \alpha$ and let $\{e'_2, \dots, e'_n\}$ be an orthonormal basis of H_α . The corresponding coordinates are denoted $y = (y_1, \dots, y_n)$. It is easy to see that $\langle \alpha, x \rangle = |\alpha| y_1$ and $\Delta = \sum_{i=1}^n \partial_{y_i}^2$. In terms of these new coordinates, L and P are expressed as

$$\begin{aligned} L &= - \sum_{i=1}^n \partial_{y_i}^2 + \frac{\langle \alpha, \alpha \rangle^{-1} C_\alpha}{y_1^2} + S(y), \\ P &= \sum_{k=0}^{m_0} P_k, \quad P_k = \sum_{|p|=m_0-k} b_p(y) \partial_y^p, \end{aligned} \quad (2.1)$$

where $\partial_y = (\partial_{y_1}, \dots, \partial_{y_n})$, and $S(y)$ is a real analytic function on $D = \{|y| < \epsilon\} \setminus \cup_{\substack{\beta \in \mathcal{H} \\ \beta \neq \alpha}} H_\beta$ for some $\epsilon > 0$. Next, let $\eta = (\eta_1, \dots, \eta_n)$ be the symbol corresponding to $\partial_y = (\partial_{y_1}, \dots, \partial_{y_n})$. Thus, we denote the symbol of P_k given in (2.1) by

$$\tilde{P}_k(y, \eta) = \sum_{|p|=m_0-k} b_p(y) \eta^p.$$

By the Leibniz rule, we have

$$(P_k \circ y_1^{-2})^\sim = \sum_{l=0}^{m_0-k} (-1)^l (l+1) y_1^{-l-2} \partial_{\eta_1}^l \tilde{P}_k.$$

Therefore, we obtain

$$\begin{aligned} [P, y_1^{-2}]^\sim &= \sum_{k=0}^{m_0-1} \sum_{l=1}^{m_0-k} (-1)^l (l+1) y_1^{-l-2} \partial_{\eta_1}^l \tilde{P}_k \\ &= \sum_{k=1}^{m_0} \sum_{l=1}^k (-1)^l (l+1) y_1^{-l-2} \partial_{\eta_1}^l \tilde{P}_{k-l}. \end{aligned}$$

On the other hand, because

$$[\Delta, P]^\sim = 2 \sum_{k=0}^{m_0} \langle \eta, \partial_y \rangle \tilde{P}_k + \sum_{k=0}^{m_0} \Delta \tilde{P}_k \quad (\langle \eta, \partial_y \rangle = \sum_{i=1}^n \eta_i \partial_{y_i}),$$

we have

$$\begin{aligned} [L, P] &= 0 \\ \Leftrightarrow 2 \sum_{k=0}^{m_0} \langle \eta, \partial_y \rangle \tilde{P}_k + \sum_{k=0}^{m_0} \Delta \tilde{P}_k \\ &\quad + \sum_{k=1}^{m_0} \sum_{l=1}^k (-1)^l (l+1) \langle \alpha, \alpha \rangle^{-1} C_\alpha y_1^{-l-2} \partial_{\eta_1}^l \tilde{P}_{k-l} + [P, S(y)]^\sim = 0 \\ \Leftrightarrow 2 \langle \eta, \partial_y \rangle \tilde{P}_{k+1} + \Delta \tilde{P}_k \\ &\quad + \sum_{l=1}^k (-1)^l (l+1) \langle \alpha, \alpha \rangle^{-1} C_\alpha y_1^{-l-2} \partial_{\eta_1}^l \tilde{P}_{k-l} \\ &\quad + (\text{the } (m_0 - k)\text{-th order terms of } [P, S(y)]^\sim = 0 \end{aligned} \quad (2.2)$$

for $k = 0, \dots, m_0$. Here, we have set $\tilde{P}_{-1} = \tilde{P}_{m_0+1} = 0$.

Lemma 2.1 *As a function of y_1 , the order of the pole of \tilde{P}_k at $y_1 = 0$ is at most k .*

PROOF. Denote by $O(F(y, \eta))$ the order of the pole of a function $F(y, \eta)$ at $y_1 = 0$. We prove this lemma by induction on k .

By assumption, \tilde{P}_0 is constant in y . Therefore, $O(\tilde{P}_0) = 0$. Now assume that $O(\tilde{P}_l) \leq l$ for $l = 0, 1, \dots, k$. Then, $O(\Delta \tilde{P}_k)$ and $O(y_1^{-l-2} \partial_{\eta_1}^l \tilde{P}_{k-l})$ are no greater than $k+2$. The $(m_0 - k)$ -th order terms of $[P, S(y)]$ come from $[P_l, S(y)]$ ($l = 0, 1, \dots, k-1$). Because $S(y)$ is real analytic at $y_1 = 0$, $O([P_l, S(y)])$ is no greater than $k-1$, by the hypothesis of induction. Hence $O(\langle \eta, \partial_y \rangle \tilde{P}_{k+1})$ is no greater than $k+2$ by (2.2), and thus the lemma holds for $k+1$. \square

Next, let $y' = (y_2, \dots, y_n)$ and $\eta' = (\eta_2, \dots, \eta_n)$, and let

$$\tilde{Q}_k(y', \eta_1, \eta') = \lim_{y_1 \rightarrow 0} y_1^k \tilde{P}_k \quad \text{and} \quad \tilde{Q}'_k(y_1, y', \eta_1, \eta') = \tilde{P}_k - y_1^{-k} \tilde{Q}_k.$$

Then, after substituting $\tilde{P}_k = y_1^{-k} \tilde{Q}_k + \tilde{Q}'_k$ into (2.2), and taking the limit $\lim_{y_1 \rightarrow 0} (y_1^{k+2} \times (2.2))$, we have

$$-2(k+1)\eta_1 \tilde{Q}_{k+1} + k(k+1) \tilde{Q}_k + \sum_{l=1}^k (-1)^l (l+1) \langle \alpha, \alpha \rangle^{-1} C_\alpha \partial_{\eta_1}^l \tilde{Q}_{k-l} = 0 \quad (2.3)$$

for $k = 0, 1, \dots, m_0$. This condition can be easily rephrased as follows.

Lemma 2.2 *The polynomials \tilde{Q}_k ($k = 0, 1, \dots, m_0+1$) satisfy (2.3) if and only if they satisfy*

$$\left[-\frac{d^2}{dt^2} + \frac{\langle \alpha, \alpha \rangle^{-1} C_\alpha}{t^2}, \sum_{k=0}^{m_0} t^{-k} \tilde{Q}_k \left(y', \frac{d}{dt}, \eta' \right) \right] = 0. \quad (2.4)$$

With this lemma, the proof of Theorem 1.1 is reduced to that for the rank-one rational case.

3 The rank-one rational case

In this section, we solve (2.4) following Burchall and Chaundy [1].

Let L_1 be a one variable Schrödinger operator:

$$L_1 = -\frac{d^2}{dt^2} + u(t).$$

Proposition 3.1 ([1]) *Assume that a differential operator A_m of order $2m+1$ with a constant principal symbol commutes with L_1 . Then A_m can be expressed as*

$$A_m = \sum_{k=0}^m \left(p_k \frac{d}{dt} - \frac{1}{2} p'_k \right) L_1^{m-k} \mod \mathcal{C}[L_1],$$

where $\{p_j; j = 0, \dots, m+1\}$ is a solution of the system of functional equations

$$\begin{cases} -\frac{1}{2} p_j''' + 2up'_j - 2p'_{j+1} + u'p_j = 0 & (j = 0, 1, \dots, m), \\ p'_0 = 0, \\ p_{m+1} = 0. \end{cases} \quad (3.1)$$

Lemma 3.2 *If the above operator A_m commutes with*

$$L_1 = -\frac{d^2}{dt^2} + \frac{\langle \alpha, \alpha \rangle^{-1} C_\alpha}{t^2},$$

then there exists $k \in \{0, 1, \dots, m\}$ such that

$$C_\alpha = k(k+1) \langle \alpha, \alpha \rangle.$$

PROOF. First, we prove that the solution of (3.1) can be expressed as

$$p_j = \sum_{i=0}^j c_{j,i} t^{-2i}, \quad (3.2)$$

with suitable constants $c_{j,i}$, by induction on j .

Because (3.1) is linear in $\{p_j\}$ and $p'_0 = 0$, we may set $c_{0,0} = 1$. Suppose that p_0, \dots, p_j are expressed as (3.2). Then (3.1) implies

$$\begin{aligned} p'_{j+1} &= -\frac{1}{4} \sum_{i=0}^j c_{j,i} (-2i)(-2i-1)(-2i-2) t^{-2i-3} \\ &\quad + \langle \alpha, \alpha \rangle^{-1} C_\alpha t^{-2} \sum_{i=0}^j c_{j,i} (-2i) t^{-2i-1} - \langle \alpha, \alpha \rangle^{-1} C_\alpha t^{-3} \sum_{i=0}^j c_{j,i} t^{-2i} \\ &= \sum_{i=0}^j (2i+1) \{i(i+1) - \langle \alpha, \alpha \rangle^{-1} C_\alpha\} c_{j,i} t^{-2i-3}. \end{aligned}$$

Therefore, if we set

$$c_{j+1,i+1} = \frac{2i+1}{2i+2} \{ \langle \alpha, \alpha \rangle^{-1} C_\alpha - i(i+1) \} c_{j,i}, \quad (3.3)$$

with $c_{j+1,0}$ arbitrary for $j \geq 0$, then p_{j+1} is also expressed as (3.2).

Now, by (3.3), we have

$$\begin{aligned} c_{m+1,m+1} &= \frac{2m+1}{2m+2} \{ \langle \alpha, \alpha \rangle^{-1} C_\alpha - m(m+1) \} c_{m,m} \\ &= \prod_{k=0}^m \frac{2k+1}{2k+2} \{ \langle \alpha, \alpha \rangle^{-1} C_\alpha - k(k+1) \}. \end{aligned}$$

If $\{p_j\}$ is a solution of (3.1), $c_{m+1,m+1}$ must be zero. Thus $C_\alpha = k(k+1)\langle \alpha, \alpha \rangle$ for some $k \in \{0, 1, \dots, m\}$. \square

Now, we return to the proof of Theorem 1.1.

Note that $\tilde{P}_0 = \tilde{Q}_0$ because \tilde{P}_0 is constant in y . Then, because $C_\alpha \neq k(k+1)\langle \alpha, \alpha \rangle$ for any $k \in \mathbf{Z}$, Lemma 3.2 and Lemma 2.2 imply that $\tilde{P}_0 = \tilde{Q}_0$ is even in η_1 ; that is, it can be expressed as $\tilde{P}_0 = \sum_{k=0}^{\lfloor m_0/2 \rfloor} \eta_1^{2k} \tilde{P}_{0,k}(\eta')$. Moreover, $\tilde{P}_{0,k}(\eta')$ is also r_α -invariant, because η_2, \dots, η_n are the symbols of directional differentials along H_α . Therefore, \tilde{P}_0 is r_α -invariant. \square

4 Hyperplane arrangement in the generic case

In this section, we address the problem of determining the permissible kinds of hyperplane arrangements when the potential function is generic, i.e. in the case that $C_\alpha \neq k(k+1)\langle \alpha, \alpha \rangle$ for any integer k and any $\alpha \in \mathcal{H}$. In order to exclude trivial cases, we assume that the principal symbol of P is not a polynomial in $\sum_{i=1}^n \xi_i^2$. Moreover, in order to avoid the possibility of reduction to a lower-dimensional case, we assume the ‘‘irreducibility’’ of the hyperplane arrangement \mathcal{H} , as defined below.

Definition 4.1 A finite subset \mathcal{H} of mutually non-parallel vectors in \mathbf{R}^n is *irreducible* if it satisfies the following conditions:

- (I1) \mathcal{H} spans \mathbf{R}^n as an \mathbf{R} -vector space.
- (I2) \mathcal{H} cannot be partitioned into the union of two proper subsets such that each vector in one subset is orthogonal to each vector in the other.

Let W be the reflection group generated by $\{r_\alpha; \alpha \in \mathcal{H}\}$ and \overline{W} be the closure of W in $O(n)$.

Proposition 4.2 If W is an infinite group, then \overline{W} is isomorphic to $O(n)$.

PROOF. By a general theory of topological groups, \overline{W} is a closed subgroup of $O(n)$, or in other words, a compact Lie subgroup.

Because $\#W = \infty$, \overline{W} contains a subgroup T isomorphic to $SO(2)$. Let $V^T = \{v \in \mathbf{R}^n; tv = v, \forall t \in T\}$. If $\mathcal{H} \subset V^T$, then T acts trivially on \mathbf{R}^n , by

(I1). This contradicts the relation $T \simeq SO(2) \subset O(n)$. Therefore, there exists $\alpha \in \mathcal{H}$ such that $t\alpha \neq \alpha$ for any $t \in T$ sufficiently close to e . We can choose $t \in T$ such that the closure of $\langle r_\alpha, tr_\alpha t^{-1} = r_{t\alpha} \rangle$ is isomorphic to $O(2)$. Let $\alpha_1 = \alpha$, $\alpha_2 = t\alpha$, $V_2 = \mathbf{R}\alpha_1 + \mathbf{R}\alpha_2$, $U_2 = V_2^\perp$ and $G_2 = \overline{\langle r_{\alpha_1}, r_{\alpha_2} \rangle}$. Then, obviously, $\mathbf{R}^n = V_2 \oplus U_2$, and G_2 is a closed subgroup of \overline{W} . Because G_2 acts trivially on U_2 , we have $G_2 \simeq \begin{pmatrix} O(2) & O \\ O & I_{n-2} \end{pmatrix} \hookrightarrow O(n)$.

Now, let us define a k -dimensional subspace V_k , the orthogonal complement U_k of V_k , and a compact subgroup G_k of \overline{W} inductively as follows. For $k < n$, not all vectors in \mathcal{H} are contained in V_k , by (I1). Therefore, $\mathcal{H}_k := \mathcal{H} \setminus (\mathcal{H} \cap V_k)$ is not empty. Since $\mathcal{H}_k \not\subset U_k$, by (I2), we can choose a vector $\alpha_{k+1} \in \mathcal{H}_k$ satisfying $\alpha_{k+1} \notin U_k$. Then, let $V_{k+1} = \mathbf{R}\alpha_{k+1} + V_k$, $U_{k+1} = \alpha_{k+1}^\perp \cap U_k = V_{k+1}^\perp$, and G_k be the closure of the group generated by G_k and $r_{\alpha_{k+1}}$. Clearly, V_{k+1} is a G_{k+1} -invariant subspace, and G_{k+1} acts trivially on U_{k+1} .

Next, choose an orthonormal basis $\{f_1, \dots, f_n\}$ of \mathbf{R}^n such that $\{f_1, \dots, f_k\}$ is a basis of V_k . By induction on k , we now show that the realization of G_k with respect to this basis is $\begin{pmatrix} O(k) & O \\ O & I_{n-k} \end{pmatrix}$.

The case $k = 2$ has been demonstrated.

Now, assume G_k to be realized as above. Then, denote by $\mathbf{a} = \begin{pmatrix} \mathbf{a}' \\ a_{k+1} \\ \mathbf{0} \end{pmatrix}$

($\mathbf{a}' \in \mathbf{R}^k$) the coordinates of α_{k+1} with respect to $\{f_i\}$. Note that $\mathbf{a}' \neq \mathbf{0}$ and $a_{k+1} \neq 0$, because $\alpha_{k+1} \notin V_{k+1} \cap U_k$ and $\alpha_{k+1} \notin V_k$. By the assumption of induction, the Lie algebra of G_k is realized as $\begin{pmatrix} \mathfrak{o}(k) & O \\ O & O \end{pmatrix}$. Let $X = \begin{pmatrix} X' & O \\ O & O \end{pmatrix}$ ($X' \in \mathfrak{o}(k)$) be an element of $\text{Lie}(G_k)$. Because the representation matrix of $r_{\alpha_{k+1}}$ is $I - 2|\mathbf{a}|^{-2}\mathbf{a}^t\mathbf{a}$, we have

$$\begin{aligned} \text{Ad}(r_{\alpha_{k+1}})X &= (I - 2|\mathbf{a}|^{-2}\mathbf{a}^t\mathbf{a})X(I - 2|\mathbf{a}|^{-2}\mathbf{a}^t\mathbf{a}) \\ &= \begin{pmatrix} X' - 2|\mathbf{a}|^{-2}(\mathbf{a}'^t\mathbf{a}'X' + X'\mathbf{a}'^t\mathbf{a}') & -2|\mathbf{a}|^{-2}a_{k+1}X'^t\mathbf{a}' & O \\ -2|\mathbf{a}|^{-2}a_{k+1}^t\mathbf{a}'X' & 0 & O \\ O & O & O \end{pmatrix}. \end{aligned}$$

Here, we have used ${}^t\mathbf{a}'X'\mathbf{a}' = 0$, as ${}^t\mathbf{a}'X'\mathbf{a}'$ is a 1×1 -alternative matrix. Since $\mathbf{a}' \neq \mathbf{0}$ and $a_{k+1} \neq 0$, there exists an $X' \in \mathfrak{o}(k)$ such that

$$\begin{pmatrix} O & -2|\mathbf{a}|^{-2}a_{k+1}X'^t\mathbf{a}' \\ -2|\mathbf{a}|^{-2}a_{k+1}^t\mathbf{a}'X' & 0 \end{pmatrix} \neq O.$$

As a Lie algebra, $\mathfrak{o}(k+1)$ is generated by this matrix and $\mathfrak{o}(k)$. Therefore, $\text{Lie}(G_{k+1})$ is realized as $\begin{pmatrix} \mathfrak{o}(k+1) & O \\ O & O \end{pmatrix}$, and G_{k+1} as $\begin{pmatrix} O(k+1) & O \\ O & I_{n-k-1} \end{pmatrix}$, because G_{k+1} acts trivially on U_{k+1} .

Finally, $O(n) = G_n \subset \overline{W}$ implies $\overline{W} = O(n)$. \square

Corollary 4.3 *If \mathcal{H} is irreducible and W is an infinite group, then any W -invariant polynomial in $\mathbf{C}[\mathbf{R}^n]$ is a polynomial in $\sum_{i=1}^n \xi_i^2$.*

Theorem 4.4 *Suppose that the principal symbol of P is constant in x and is not a polynomial in $\sum_{i=1}^n \xi_i^2$. Then, if $R(x)$ in (1.2) is generic and \mathcal{H} is irreducible, either W is a finite reflection group or $[L, P] \neq 0$.*

PROOF. Assume that L and P are commutative. Then, by Theorem 1.1, the principal symbol \tilde{P}_0 of P is a W -invariant polynomial. If W is infinite, \tilde{P}_0 must be a polynomial in $\sum_{i=1}^n \xi_i^2$, by Corollary 4.3. However, this contradicts the assumption. Therefore W is a finite reflection group. \square

By this theorem, in generic cases, we need consider only the case in which \mathcal{H} is a subset of the root system of a finite reflection group.

5 Determination of the potential – general situation

Assume that \mathcal{H} is irreducible and that $R(x)$ is generic. Then, as stated above, we need consider only the case in which $W = \langle r_\alpha; \alpha \in \mathcal{H} \rangle$ is a finite reflection group; that is, we may regard \mathcal{H} as a subset of the positive root system Σ^+ of W . In subsequent sections, we determine the potential function $R(x)$ in the cases that the root system Σ is of type A , B and D under some conditions. In this section, we explain the general situation.

Let P be a commutant of L with a constant principal symbol. Because $R(x)$ is generic, the principal symbol of P is W -invariant, by Theorem 1.1. We assume the following conditions:

- (1) P is real analytic in the domain where L is defined.
- (2) The order of P is the smallest degree of W larger than 2.

In general, for a differential operator $D = \sum_p a_p(x) \partial^p$, we define ${}^t D$ as

$${}^t D = \sum_p (-1)^{|p|} \partial^p \circ a_p(x),$$

and call it the adjoint operator of P . Because L is self-adjoint (i.e. ${}^t L = L$), if P commutes with L , so does ${}^t P$. Therefore, we may assume that P is (skew-) self-adjoint, i.e. ${}^t P = (-1)^{\text{ord} P} P$.

6 Determination of the potential – type A_{n-1}

The arguments hereafter are quite similar to those in [10]. There, the Weyl group invariance of L and P is assumed, but here this assumption is not made. This is the most important difference between the situations considered here and in that work.

The root system of type A_{n-1} is realized in the hyperplane

$$V = \left\{ (x_1, \dots, x_n) \in \mathbf{R}^n; \sum_{i=1}^n x_i = 0 \right\},$$

and we choose a positive system as

$$\Sigma^+ = \{e_i - e_j; 1 \leq i < j \leq n\}.$$

By virtue of this realization, the Schrödinger operator (1.2) is extended to the operator

$$L = -\Delta + R(x), \quad R(x) = \sum_{1 \leq i < j \leq n} \frac{C_{ij}}{(x_i - x_j)^2} + \tilde{R}(x), \quad (6.1)$$

defined on some open subset of \mathbf{R}^n , where $\tilde{R}(x)$ is a real analytic at $x = 0$ and L commutes with

$$\Delta_1 = \sum_{i=1}^n \partial_i.$$

Note that some of the constants C_{ij} may be zero, because \mathcal{H} might not coincide with Σ^+ .

As a commutant P of L , we can choose

$$P = \sum_{1 \leq i < j < k \leq n} \partial_i \partial_j \partial_k + \sum_{i=1}^n a_1^i \partial_i + a_0, \quad (6.2)$$

which commutes with Δ_1 .

As seen from Remark 2.4 of [10], the equations $[L, P] = 0$, $[L, \Delta_1] = 0$ and $[\Delta_1, P] = 0$ imply that $R(x)$ can be expressed as

$$R(x) = \sum_{1 \leq i < j \leq n} u_{ij}(x_i - x_j) \quad (6.3)$$

with suitable functions $u_{ij}(t) = C_{ij}t^{-2} + \gamma_{ij}(t)$, where $\gamma_{ij}(t)$ is real analytic at $t = 0$. For convenience, let $u_{ij}(t) = u_{ji}(-t)$ for $j < i$.

Lemma 6.1 *We can choose a_1^i as*

$$a_1^i = \frac{1}{2} \sum_{\substack{j < k \\ \neq i}} u_{jk}(x_j - x_k), \quad (6.4)$$

because we are free to choose u_{ij} appropriately.

PROOF. The second-order terms of $[L, P] = 0$ imply

$$\partial_i^2 : \partial_i a_1^i = 0, \quad (6.5)$$

$$\partial_i \partial_j : \partial_j a_1^i + \partial_i a_1^j = -\frac{1}{2} \sum_{k \neq i, j} \partial_k R = \frac{1}{2} (\partial_i + \partial_j) R. \quad (6.6)$$

Also, by (6.6), we have

$$\partial_j \partial_k a_1^i + \partial_i \partial_k a_1^j = \frac{1}{2} \partial_k (\partial_i + \partial_j) R, \quad (6.7)$$

$$\partial_k \partial_i a_1^j + \partial_j \partial_i a_1^k = \frac{1}{2} \partial_i (\partial_j + \partial_k) R, \quad (6.8)$$

$$\partial_i \partial_j a_1^k + \partial_k \partial_j a_1^i = \frac{1}{2} \partial_j (\partial_k + \partial_i) R. \quad (6.9)$$

Taken together, (6.7), (6.9) and (6.8) imply

$$\partial_j \partial_k a_1^i = \frac{1}{2} \partial_j \partial_k R = -\frac{1}{2} u_{jk}''(x_j - x_k).$$

Moreover, because the relation $[\Delta_1, P] = 0$ implies $\Delta_1 a_1^i = 0$, we have the following:

$$\partial_j^2 a_1^i = \partial_j (\partial_j - \Delta_1) a_1^i = - \sum_{\substack{k < j \\ k \neq i}} \partial_j \partial_k a_1^i = \frac{1}{2} \sum_{k \neq i, j} u_{jk}''(x_j - x_k).$$

Therefore, we may put

$$a_1^i = \frac{1}{2} \sum_{\substack{j < k \\ j \neq i}} u_{jk}(x_j - x_k) + \sum_{j \neq i} p_{i,j} x_j + q_i.$$

Equation (6.6) implies $p_{i,j} = -p_{j,i}$, and $\Delta_1 a_1^i = 0$ implies $\sum_{j \neq i} p_{i,j} = 0$. Next, let $\overset{\circ}{q} = (\sum_{i=1}^n q_i)/n$, $\tilde{q}_i = q_i - \overset{\circ}{q}$ and

$$\tilde{u}_{ij}(t) = u_{ij}(t) + 2p_{i,j}t + \beta_{ij}$$

for $1 \leq i < j \leq n$, where the quantities β_{ij} are given by

$$\begin{aligned} \beta_{i1} &= \beta_{1i} = -2\tilde{q}_i \quad (i = 2, \dots, n-2, \text{ if } n > 3), \\ \beta_{n-1,i} &= \beta_{1,n-1} = -2(\tilde{q}_1 + \tilde{q}_{n-1}), \\ \beta_{n,1} &= \beta_{1,n} = -2(\tilde{q}_1 + \tilde{q}_n), \\ \beta_{n,n-1} &= \beta_{n-1,n} = 2\tilde{q}_1, \quad \text{and} \\ \beta_{ij} &= 0 \quad (\text{otherwise}). \end{aligned}$$

Then, because $\sum_{i=1}^n \tilde{q}_i = 0$, we have $\sum_{i < j} \beta_{ij} = -2 \sum_{i=2}^{n-2} \tilde{q}_i - 2(\tilde{q}_1 + \tilde{q}_{n-1}) - 2(\tilde{q}_1 + \tilde{q}_n) + 2\tilde{q}_1 = 0$ and $\sum_{\substack{j < k \\ j \neq i}} \beta_{ij} = 2\tilde{q}_i$. Therefore,

$$\begin{aligned} \sum_{1 \leq i < j \leq n} u_{ij}(x_i - x_j) &= \sum_{1 \leq i < j \leq n} (\tilde{u}_{ij}(x_i - x_j) - 2p_{i,j}(x_i - x_j) - \beta_{ij}) \\ &= \sum_{1 \leq i < j \leq n} \tilde{u}_{ij}(x_i - x_j) \end{aligned}$$

and

$$\begin{aligned} a_1^i &= \frac{1}{2} \sum_{\substack{j < k \\ j \neq i}} (\tilde{u}_{jk}(x_j - x_k) - 2p_{j,k}(x_j - x_k) - \beta_{jk}) + \sum_{j \neq i} p_{i,j} x_j + \tilde{q}_i + \overset{\circ}{q} \\ &= \frac{1}{2} \sum_{\substack{j < k \\ j \neq i}} \tilde{u}_{jk}(x_j - x_k) + \overset{\circ}{q}. \end{aligned}$$

Hence, by subtracting $\overset{\circ}{q}\Delta_1$ from P , we obtain (6.4). \square

The condition ${}^tP = -P$ is equivalent to

$$a_0 = \frac{1}{2} \sum_{i=1}^n \partial_i a_1^i = 0.$$

Therefore, the zeroth-order term of the relation $[L, P] = 0$ implies

$$\sum_{i < j < k} \partial_i \partial_j \partial_k R + \sum_i a_1^i \partial_i R = 0.$$

Applying (6.3) and (6.4) to this equation, we have

$$\sum_{i < j} \left(\sum_{p \neq i, j} (u_{pj}(x_p - x_j) - u_{pi}(x_p - x_i)) \right) u'_{ij}(x_i - x_j) = 0. \quad (6.10)$$

Then, because $u_{ij}(t) - C_{ij}t^{-2}$ is real analytic at $t = 0$, $\lim_{x_j \rightarrow x_i} ((x_i - x_j)^3 \times (6.10))$ gives

$$C_{ij} \sum_{p \neq i, j} (u_{pi}(x_p - x_i) - u_{pj}(x_p - x_j)) = 0. \quad (6.11)$$

Moreover, $\lim_{x_k \rightarrow x_i} ((x_k - x_i)^2 \times (6.11))$ gives

$$C_{ij}(C_{ki} - C_{kj}) = 0 \quad (6.12)$$

for $k \neq i, j$. Because \mathcal{H} is not empty, there exist i_1 and i_2 ($i_1 \neq i_2$) such that $C_{i_1 i_2} \neq 0$. Then, employing an appropriate coordinate transformation, we can put $i_1 = 1$ and $i_2 = 2$. Therefore, by (6.12), we have $C_{1i} = C_{2i}$ for $i \geq 3$. The condition (I2) and the relation $C_{12} \neq 0$ imply that there exists i_3 such that $C_{1i_3} = C_{2i_3} \neq 0$. Again, our ability to apply coordinate transformations allows us to choose $i_3 = 3$. Then, from (6.12), we find $C_{12} = C_{23} = C_{13}$ and $C_{1i} = C_{3i} = C_{2i}$ for $i \geq 4$. In the same way, we can show inductively that C_{ij} depends on neither i nor j . In particular, none of them are zero.

The fact that $C_{ij} \neq 0$ and equation (6.11) together imply

$$u'_{ki}(t) = u'_{kj}(t). \quad (6.13)$$

Then, because $u_{ij}(t) = u_{ji}(-t)$, (6.13) implies $u'_{ik}(t) = u'_{jk}(t)$, and we have

$$u'_{ij}(t) = u'_{ik}(t) = u'_{jk}(t) = -u'_{kj}(-t) = -u'_{ij}(-t).$$

Therefore $u_{ij}(t)$ is an even function and, by (6.13), there exist constants c_{ij} ($1 \leq i < j \leq n$) and an even function $u(t)$ such that

$$u_{ij}(t) = u(t) + c_{ij}.$$

Because $u(t)$ is fixed only up to an arbitrary constant, we can choose the c_{ij} so that $\sum_{i < j} c_{ij} = 0$. From (6.11), we obtain $\sum_{p \neq i, j} c_{pi} = \sum_{p \neq i, j} c_{pj} \Leftrightarrow \sum_{p \neq i} c_{pi} = \sum_{p \neq j} c_{pj}$. This means that $\tilde{c} = \sum_{p \neq i} c_{pi}$ does not depend on i . Then, because $\sum_{i < j} c_{ij} = 0$, we have

$$R(x) = \sum_{1 \leq i < j \leq n} u_{ij}(x_i - x_j) = \sum_{1 \leq i < j \leq n} (u(x_i - x_j) + c_{ij}) = \sum_{1 \leq i < j \leq n} u(x_i - x_j)$$

and

$$a_1^i = \frac{1}{2} \sum_{\substack{j < k \\ \neq i}} u_{jk}(x_j - x_k) = \frac{1}{2} \sum_{\substack{j < k \\ \neq i}} (u(x_j - x_k) + c_{jk}) = \frac{1}{2} \sum_{\substack{j < k \\ \neq i}} u(x_j - x_k) - \frac{\tilde{c}}{2}.$$

Then, the freedom we have to add $(\tilde{c}/2)\Delta_1$ to P allows us to realize the condition $c_{ij} = 0$ for all $i \neq j$. In this case, (6.10) becomes

$$\sum_{i < j} \left(\sum_{p \neq i, j} (u(x_p - x_j) - u(x_p - x_i)) \right) u'(x_i - x_j) = 0.$$

In [10], Oshima and Sekiguchi solved this functional equation. They obtained the solution

$$u(t) = c_1 \wp(t|2\omega_1, 2\omega_2) + c_2,$$

where c_1 and c_2 are arbitrary constants and $\wp(t|2\omega_1, 2\omega_2)$ is the Weierstrass elliptic function with primitive periods $2\omega_1$ and $2\omega_2$.

Combining the above results, we have proved the following theorem.

Theorem 6.2 *If L in (6.1) commutes with P in (6.2), then there exist constants c_1 and c_2 such that*

$$R(x) = c_1 \sum_{1 \leq i < j \leq n} \wp(x_i - x_j|2\omega_1, 2\omega_2) + c_2.$$

7 Determination of the potential – types B_n and D_n

Assume W to be of type B_n ($n \geq 2$) or D_n ($n \geq 4$). The root systems of type B_n and D_n are realized in \mathbf{R}^n . We choose

$$\begin{aligned} \Sigma^+ &= \{e_i \pm e_j; 1 \leq i < j \leq n\} \cup \{e_i; 1 \leq i \leq n\} && \text{for } B_n\text{-type and} \\ \Sigma^+ &= \{e_i \pm e_j; 1 \leq i < j \leq n\} && \text{for } D_n\text{-type} \end{aligned}$$

as their positive systems. In these cases, the Schrödinger operator (1.2) is

$$L = -\Delta + R(x),$$

$$R(x) = \sum_{1 \leq i < j \leq n} \left(\frac{C_{ij}^+}{(x_i + x_j)^2} + \frac{C_{ij}^-}{(x_i - x_j)^2} \right) + \sum_{i=1}^n \frac{C_i}{x_i^2} + \tilde{R}(x), \quad (7.1)$$

where $\tilde{R}(x)$ is real analytic at $x = 0$, and $C_i = 0$ for $i = 1, \dots, n$ in the D_n case.

As the commutant P satisfying the two conditions in §5, we can choose

$$P = \sum_{1 \leq i < j \leq n} \partial_i^2 \partial_j^2 + \sum_{i=1}^n a_2^i \partial_i^2 + \sum_{1 \leq i < j \leq n} a_{11}^{ij} \partial_i \partial_j + \sum_{i=1}^n a_1^i \partial_i + a_0. \quad (7.2)$$

For convenience, we set $a_{11}^{ij} = a_{11}^{ji}$ for $j < i$.

Remark 7.1 In the D_4 case, other choices of P are possible, since

$$P = c_1 \sum_{1 \leq i < j \leq 4} \partial_i^2 \partial_j^2 + c_2 \partial_1 \partial_2 \partial_3 \partial_4 + (\text{lower-order terms})$$

satisfies the two conditions in §5 for any c_1 and c_2 . If $c_1 = 1$ and $c_2 = \pm 6$, the fourth order term of P changes to

$$\frac{3}{4} \sum_{i=1}^4 \partial_{y_i^\pm}^4 - \frac{1}{2} \sum_{1 \leq i < j \leq 4} \partial_{y_i^\pm}^2 \partial_{y_j^\pm}^2$$

through the orthogonal coordinate transformation defined by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} := \frac{1}{2} \begin{pmatrix} 1 & \pm 1 & \pm 1 & \pm 1 \\ \pm 1 & 1 & -1 & -1 \\ \pm 1 & -1 & 1 & -1 \\ \pm 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

Therefore, the situation regarding the commutator of this operator and L is equivalent to that for the operator in (7.2) and L . In the case $c_2 \neq \pm 6c_1$, it can be shown that $R(x)$ can be expressed as

$$\sum_{1 \leq i < j \leq 4} (u(x_i - x_j) + u(x_1 + x_j)) + \sum_{i=1}^4 v(x_i), \quad (7.3)$$

with $u(t)$ and $v(t)$ appropriately chosen even functions. This is identical to the assertion of Theorem 7.7 below.

Now, let us return to the situation described prior to Remark 7.1. The third-order terms of the relation $[L, P] = 0$ imply

$$\partial_i^3 : \quad \partial_i a_2^i = 0, \quad (7.4)$$

$$\partial_i^2 \partial_j : \quad \partial_j a_2^i + \partial_i a_{11}^{ij} = -\partial_j R, \quad (7.5)$$

$$\partial_i \partial_j \partial_k : \quad \partial_k a_{11}^{ij} + \partial_i a_{11}^{jk} + \partial_j a_{11}^{ik} = 0. \quad (7.6)$$

Lemma 7.2 ([10, Lemma 2.5])

(1) Let $n \geq 3$. If the functions $u_i(x)$ ($1 \leq i \leq n$) and $u_{ij}(x) = u_{ji}(x)$ ($1 \leq i < j \leq n$) of $x = (x_1, \dots, x_n)$ satisfy

$$\partial_j u_i + \partial_i u_{ij} = 0 \quad \text{and} \quad \partial_k u_{ij} + \partial_i u_{jk} + \partial_j u_{ik} = 0, \quad (7.7)$$

then

$$\partial_j^2 \partial_k u_i = 0 \quad \text{and} \quad \partial_j \partial_k \partial_l u_i = 0.$$

(2) Moreover, if they also satisfy

$$\partial_i u_i = 0 \quad (7.8)$$

for $1 \leq i \leq n$, then

$$\partial_j^2 u_{ij} = 0, \quad \text{and} \quad \partial^\alpha u_i = \partial^\alpha u_{ij} = 0 \quad \text{if } |\alpha| \geq 3. \quad (7.9)$$

(3) If $n = 2$, the first relation in (7.7) and the relation (7.8) imply (7.9).

First, assume that $n \geq 3$, and let $u_i = a_2^i + R$ and $u_{ij} = a_{11}^{ij}$. Then, the relations (7.5) and (7.6) imply that u_i and u_{ij} satisfy the conditions in Lemma 7.2 (1). Therefore, $\partial_j^2 \partial_k (a_2^i + R) = 0$ and $\partial_j \partial_k \partial_l (a_2^i + R) = 0$, which imply $\partial_i \partial_j^2 \partial_k R = \partial_i \partial_j \partial_k \partial_l R = 0$. Therefore, $\partial_i \partial_j \partial_k R$ is a constant, and $\partial_i \partial_j R$ can be expressed as

$$\partial_i \partial_j R = \sum_{k \neq i, j} c_{ijk} x_k + \phi_{ij}(x_i, x_j),$$

for an appropriate choice of the constants c_{ijk} and the function $\phi_{ij}(x_i, x_j)$. Note that this expression is also valid for the B_2 case, in which the first term of it is ignored. Now, from the relations (7.4) and (7.5), we have $\partial_i \partial_j (\partial_i^2 - \partial_j^2) R = \partial_i \partial_j \{ \partial_i^2 (R + a_2^i) - \partial_j^2 (R + a_2^j) \} = \partial_i \partial_j (\partial_i \partial_j a_{11}^{ij} - \partial_i \partial_j a_{11}^{ij}) = 0$. Hence $(\partial_i^2 - \partial_j^2) \phi_{ij} = 0$. It follows that $\phi_{ij}(x_i, x_j) = u_{ij}^{+''}(x_i + x_j) - u_{ij}^{-''}(x_i - x_j)$, with suitable functions $u_{ij}^{\pm}(t) = u_{ji}^{\pm}(\pm t)$.

Now, let

$$\bar{R} = R - \sum_{1 \leq i < j \leq n} (u_{ij}^{+}(x_i + x_j) + u_{ij}^{-}(x_i - x_j)) - \sum_{1 \leq i < j < k \leq n} c_{ijk} x_i x_j x_k.$$

This function satisfies the relation $\partial_i \partial_j \bar{R} = 0$ for any $i < j$. This implies that $\bar{R}(x)$ is a sum of one variable functions in x_i ($1 \leq i \leq n$). Thus, we have proved the following lemma.

Lemma 7.3 *There exist constants c_{ijk} and one variable functions u_{ij}^{\pm} and v_i such that*

$$\begin{aligned} R(x) = & \sum_{1 \leq i < j \leq n} (u_{ij}^{+}(x_i + x_j) + u_{ij}^{-}(x_i - x_j)) + \sum_{i=1}^n v_i(x_i) \\ & + \sum_{1 \leq i < j < k \leq n} c_{ijk} x_i x_j x_k. \end{aligned}$$

If $n = 2$, the last term is ignored.

Note that we may assume $u_{ij}^{\pm}(t) - C_{ij}^{\pm} t^{-2}$ and $v_i(t) - C_i t^{-2}$ to be real analytic at $t = 0$, as $R(x)$ is given by (7.1).

Let \tilde{a}_2^i and \tilde{a}_{11}^{ij} be functions defined as

$$\begin{aligned} a_2^i &= \tilde{a}_2^i - \sum_{\substack{j < k \\ \neq i}} \{ u_{jk}^{+}(x_j + x_k) + u_{jk}^{-}(x_j - x_k) \} - \sum_{j \neq i} v_j(x_j) \\ &\quad - \sum_{\substack{j < k < l \\ \neq i}} c_{jkl} x_j x_k x_l, \\ a_{11}^{ij} &= \tilde{a}_{11}^{ij} - u_{ij}^{+}(x_i + x_j) + u_{ij}^{-}(x_i - x_j) \\ &\quad - \frac{1}{2}(x_i^2 + x_j^2) \sum_{k \neq i, j} c_{ijk} x_k + \frac{1}{3} \sum_{k \neq i, j} c_{ijk} x_k^3. \end{aligned}$$

We can easily show that \tilde{a}_2^i and \tilde{a}_{11}^{ij} satisfy the conditions in Lemma 7.2 (2).

Now, note that the condition ${}^t P = P$ is equivalent to

$$\begin{aligned} a_1^i &= \partial_i a_2^i + \frac{1}{2} \sum_{j \neq i} \partial_j a_{11}^{ij} \\ &= -\frac{1}{2} \sum_{j \neq i} \{ u_{ij}^{+'}(x_i + x_j) + u_{ij}^{-'}(x_i - x_j) - \partial_j \tilde{a}_{11}^{ij} \} - \sum_{\substack{j < k \\ \neq i}} c_{ijk} x_j x_k. \end{aligned} \quad (7.10)$$

Next, the coefficient of ∂_i in the relation $[L, P] = 0$ implies

$$-2\partial_i a_0 = 2 \sum_{j \neq i} \partial_i \partial_j^2 R + 2a_2^i \partial_i R + \sum_{j \neq i} a_{11}^{ij} \partial_j R + \Delta a_1^i.$$

Using this, we find that the compatibility condition $\partial_j(\partial_i a_0) = \partial_i(\partial_j a_0)$ is equivalent to

$$3(\partial_j a_{11}^{ij} \partial_j R - \partial_i a_{11}^{ij} \partial_i R) + 2(a_2^i - a_2^j) \partial_i \partial_j R + a_{11}^{ij} (\partial_j^2 - \partial_i^2) R \quad (7.11)$$

$$+ \sum_{k \neq i, j} (\partial_j a_{11}^{ik} - \partial_i a_{11}^{jk}) \partial_k R + \sum_{k \neq i, j} (a_{11}^{ik} \partial_j \partial_k R - a_{11}^{jk} \partial_i \partial_k R) = 0.$$

Here, we have used (7.5) and the relations $\partial_i \partial_j (\partial_i^2 - \partial_j^2) R = 0$ and $\Delta(\partial_i a_1^j - \partial_j a_1^i) = 0$. The last of these is a consequence of (7.10). From (7.1), it is seen that only the term $2(a_2^i - a_2^j) \partial_i \partial_j R$ can have poles of order four at $x_i \pm x_j = 0$. Therefore, taking $\lim_{x_j \rightarrow \mp x_i} ((x_i \pm x_j)^4 \times (7.11))$, we obtain

$$C_{ij}^- \left\{ \sum_{k \neq i, j} (u_{ik}^+(x_i + x_k) + u_{ik}^-(x_i - x_k) - u_{jk}^+(x_i + x_k) - u_{jk}^-(x_i - x_k)) \right. \quad (7.12)$$

$$\left. + v_i(x_i) - v_j(x_i) + x_i \sum_{\substack{k < l \\ \neq i, j}} (c_{ikl} - c_{jkl}) x_k x_l + (\tilde{a}_2^i - \tilde{a}_2^j)|_{x_j=x_i} \right\} = 0,$$

and

$$C_{ij}^+ \left\{ \sum_{k \neq i, j} (u_{ik}^+(x_i + x_k) + u_{ik}^-(x_i - x_k) - u_{jk}^+(-x_i + x_k) - u_{jk}^-(-x_i - x_k)) \right. \quad (7.13)$$

$$\left. + v_i(x_i) - v_j(-x_i) + x_i \sum_{\substack{k < l \\ \neq i, j}} (c_{ikl} + c_{jkl}) x_k x_l + (\tilde{a}_2^i - \tilde{a}_2^j)|_{x_j=-x_i} \right\} = 0.$$

Moreover, because only the terms $a_{11}^{ij} \partial_j^2 R$ and $a_{11}^{ik} \partial_j \partial_k R$ can have poles of order four at $x_j \pm x_k = 0$, taking $\lim_{x_k \rightarrow \mp x_j} ((x_j \pm x_k)^4 \times (7.11))$, we obtain

$$C_{jk}^\pm \left\{ u_{ij}^+(x_i + x_j) - u_{ij}^-(x_i - x_j) \pm u_{ik}^+(x_i \mp x_j) \mp u_{ik}^-(x_i \pm x_j) \right. \quad (7.14)$$

$$\left. + \sum_{l \neq i, j, k} (c_{ijl} \pm c_{ikl}) \left(\frac{1}{2} (x_i^2 + x_j^2) x_l - \frac{1}{3} x_l^3 \right) - (\tilde{a}_{11}^{ij} \pm \tilde{a}_{11}^{ik})|_{x_k=\mp x_j} \right\} = 0.$$

Finally, because only the term $a_{11}^{ij} \partial_i^2 R$ can have a pole of order four at $x_i = 0$, taking $\lim_{x_i \rightarrow 0} (x_i^4 \times (7.11))$, we obtain

$$C_i \left\{ u_{ij}^+(x_j) - u_{ij}^-(-x_j) + \sum_{k \neq i, j} c_{ijk} \left(\frac{1}{2} x_j^2 x_k - \frac{1}{3} x_k^3 \right) - \tilde{a}_{11}^{ij}|_{x_i=0} \right\} = 0. \quad (7.15)$$

Next, the limits $\lim_{x_k \rightarrow \mp x_i} ((x_i \pm x_k)^2 \times (7.12))$, $\lim_{x_k \rightarrow \mp x_i} ((x_i \pm x_k)^2 \times (7.13))$, $\lim_{x_i \rightarrow 0} (x_i^2 \times (7.12))$ or $(7.13))$ and $\lim_{x_j \rightarrow 0} (x_j^2 \times (7.15))$ give

$$C_{ij}^-(C_{ik}^\pm - C_{jk}^\pm) = 0, \quad (7.16)$$

$$C_{ij}^+(C_{ik}^\pm - C_{jk}^\mp) = 0, \quad (7.17)$$

$$C_{ij}^\pm(C_i - C_j) = 0, \quad (7.18)$$

$$C_i(C_{ij}^+ - C_{ij}^-) = 0 \quad (7.19)$$

for $i \neq j \neq k \neq i$, because $u_{ij}^\pm(t)$ and $v_i(t)$ can have poles of order two at $t = 0$.

Because \mathcal{H} is not empty, at least one of C_{ij}^\pm ($1 \leq i < j \leq n$) or C_i ($1 \leq i \leq n$) is not zero. If all the C_{ij}^\pm are zero, then \mathcal{H} is divided into nonempty orthogonal subsets. However, this contradicts the condition (I2). Therefore, applying an appropriate coordinate transformation, we are able to realize the condition $C_{12}^- \neq 0$. Then, (7.16) and (7.18) imply $C_{1i}^\pm = C_{2i}^\pm$ for $i \geq 3$ and $C_1 = C_2$.

If $n = 2$ and $C_1 = C_2 = 0$, then $\mathcal{H} = \{e_1 + e_2, e_1 - e_2\}$, but this contradicts the condition (I2). Therefore, $C_1 = C_2 \neq 0$. Then, from (7.19), we obtain $C_{12}^+ = C_{12}^- \neq 0$. Therefore, \mathcal{H} coincides with the positive system of the root system of type B_2 .

Now, assume $n \geq 3$. Then, using the same argument as in the A_{n-1} -case, we can show that C_{ij}^+ , C_{ij}^- and C_i are all independent of i and j . We write $C^\pm := C_{ij}^\pm$ and $C := C_i$. If $C^+ \neq C^-$, then $C = 0$ and $C^+ = 0$, as found from (7.17) and (7.19). This implies that the hyperplane arrangement \mathcal{H} is an A_{n-1} -type positive system, which contradicts our assumption $W = W(B_n)$ or $W(D_n)$. Therefore, $C^+ = C^- \neq 0$. If $C = 0$, then \mathcal{H} is of type D_n , and if $C \neq 0$ it is of type B_n .

Combining the above results, we have proved the following proposition.

Proposition 7.4 *Under the assumptions made in this section, the hyperplane arrangement \mathcal{H} coincides with the positive root system of type B_n or type D_n . Moreover, the parameters C_α in (1.2) are W -invariant.*

Lemma 7.5 *For any $1 \leq i < j < k \leq n$, we have*

$$c_{ijk} = 0.$$

PROOF. We can assume $n \geq 3$. If \mathcal{H} is of type B_n , then the fact that $C_i \neq 0$, the relation obtained by applying ∂_k^3 to (7.15), and Lemma 7.2 together imply that $c_{ijk} = 0$. If \mathcal{H} is of type D_n ($n \geq 4$), then the fact that $C_{jk}^\pm \neq 0$, the relation obtained by applying ∂_l^3 to (7.14), and Lemma 7.2 together imply that $c_{ijl} \pm c_{ikl} = 0$, and hence $c_{ijk} = 0$ for any i, j, k . \square

Lemma 7.6 *\tilde{a}_2^i and \tilde{a}_{11}^{ij} can be expressed as follows:*

$$\tilde{a}_2^i = -\frac{1}{2} \sum_{j \neq i} \alpha_{ij} x_j^2 - \sum_{j \neq i} \beta_{ij}(i) x_j + \delta_i, \quad (7.20)$$

$$\tilde{a}_{11}^{ij} = \alpha_{ij} x_i x_j + \beta_{ij}(i) x_i + \beta_{ij}(j) x_j + \gamma_{ij}. \quad (7.21)$$

Here, $\alpha_{ij}, \dots, \delta_i$ can be any constants satisfying $\sum_{i=1}^n \delta_i = 0$.

PROOF. First, we show that \tilde{a}_{11}^{ij} can be expressed as (7.21).

If \mathcal{H} is of type B_2 , this is clear from Lemma 7.2.

If \mathcal{H} is of type B_3 , we may assume that \tilde{a}_{11}^{ij} is given by

$$\tilde{a}_{11}^{ij} = \alpha_{ij} x_i x_j + \varphi_{ij}^i(x_k) x_i + \varphi_{ij}^j(x_k) x_j + \varphi_{ij}(x_k),$$

with suitable polynomials φ_{ij}^i , φ_{ij}^j and φ_{ij} , by Lemma 7.2. Then, (7.15) and Lemma 7.5 imply

$$\partial_k \tilde{a}_{11}^{ij}|_{x_i=0} = 0 \quad \text{for } k \neq i, j,$$

so that $\partial_k \varphi_{ij}^j = \partial_k \varphi_{ij} = 0$. Therefore, \tilde{a}_{11}^{ij} can be expressed as in (7.21).

If \mathcal{H} is of type B_n or D_n ($n \geq 4$), then we find

$$\partial_l (\tilde{a}_{11}^{ik} \pm \tilde{a}_{11}^{ij})|_{x_k=\mp x_j} = 0$$

from (7.14) and Lemma 7.5. Then, using the same argument as in the B_3 case, we can show that \tilde{a}_{11}^{ij} is expressed as in (7.21).

Next, note that because

$$\partial_j \tilde{a}_2^i = -\partial_i \tilde{a}_{11}^{ij} = -\alpha_{ij} x_j - \beta_{ij}(i),$$

\tilde{a}_2^i can be expressed as in (7.20). By subtracting appropriate constant multiple of L from P , we can realize the condition $\sum_{i=1}^n \delta_i = 0$. \square

Now, let

$$\begin{aligned} \tilde{u}_{ij}^\pm(t) &= u_{ij}^\pm(t) - \frac{\alpha_{ij}}{4} t^2 - \frac{\beta_{ij}(j) \pm \beta_{ij}(i)}{2} t \mp \frac{\gamma_{ij}}{2} \quad \text{and} \\ \tilde{v}_i(t) &= v_i(t) + \frac{1}{2} \left(\sum_{j \neq i} \alpha_{ij} \right) t^2 + \left(\sum_{j \neq i} \beta_{ij}(j) \right) t + \delta_i. \end{aligned}$$

Then, we have

$$\begin{aligned} R(x) &= \sum_{i < j} \{ \tilde{u}_{ij}^+(x_i + x_j) + \tilde{u}_{ij}^-(x_i - x_j) \} + \sum_i \tilde{v}_i(x_i), \\ a_2^i &= - \sum_{\substack{j < k \\ j \neq i}} \{ \tilde{u}_{jk}^+(x_j + x_k) + \tilde{u}_{jk}^-(x_j - x_k) \} - \sum_{j \neq i} \tilde{v}_j(x_j), \end{aligned} \quad (7.22)$$

$$a_{11}^{ij} = -\tilde{u}_{ij}^+(x_i + x_j) + \tilde{u}_{ij}^-(x_i - x_j). \quad (7.23)$$

Hence, we can realize the relation $\tilde{a}_2^i = \tilde{a}_{11}^{ij} = 0$ by appropriately choosing u and v .

Theorem 7.7 *There exist even functions $u(t)$ and $v(t)$ such that*

$$R(x) = \sum_{1 \leq i < j \leq n} \{ u(x_i + x_j) + u(x_i - x_j) \} + \sum_{i=1}^n v(x_i). \quad (7.24)$$

PROOF. First, we prove the assertion for the B_2 case. From (7.12) and (7.13), we have

$$v_2(t) = v_1(t) = v_2(-t),$$

which implies that $v_1 = v_2$ and that they are even functions. Then, from (7.15), we obtain

$$u_{12}^+(t) = u_{12}^-(-t) \quad \text{and} \quad u_{21}^+(t) = u_{21}^-(-t).$$

Therefore, because $u_{21}^\pm(t) = u_{12}^\pm(\pm t)$, we find that $u_{12}^+(t) = u_{12}^-(t)$, and they are even functions.

Now, consider the B_n and D_n cases for $n \geq 3$. In this case, from (7.14), we have

$$u_{ij}^+(x_i + x_j) - u_{ij}^-(x_i - x_j) \pm u_{ik}^+(x_i \mp x_j) \mp u_{ik}^-(x_i \pm x_j) = 0,$$

which imply

$$\begin{aligned} u_{ij}^+(s) - u_{ik}^-(s) &= u_{ij}^-(t) - u_{ik}^+(t), \\ u_{ik}^+(s) - u_{ik}^-(s) &= -(u_{ik}^+(t) - u_{ik}^-(t)). \end{aligned}$$

Hence, $u_{ij}^+(t) - u_{ik}^-(t)$ is a constant, and $u_{ik}^+(t) = u_{ik}^-(t)$. Therefore, there exist constants p_{ij} and a function $u(t)$ such that

$$u_{ij}^+(t) = u_{ij}^-(t) = u(t) + p_{ij}. \quad (7.25)$$

Note that $u(t)$ is an even function, because

$$u(t) - u(-t) = u_{ij}^+(t) - u_{ji}^-(t) = u_{ij}^+(t) - u_{ij}^-(t) = 0.$$

Here, we have used $u_{ij}^\pm(t) = u_{ji}^\pm(\pm t)$. Moreover, because $u(t)$ is fixed up to an arbitrary constant, we can realize the condition

$$\sum_{i < j} p_{ij} = 0. \quad (7.26)$$

Next, substituting (7.25) into (7.12), we obtain $2 \sum_{k \neq i, j} (p_{ik} - p_{jk}) + v_i(x_i) - v_j(x_j) = 0$, or

$$v_i(t) + 2 \sum_{k \neq i} p_{ik} = v_j(t) + 2 \sum_{k \neq j} p_{jk}.$$

This implies that the function $v_i(t) + 2 \sum_{k \neq i} p_{ik}$ is independent of i , and we write it as $v(t)$.

From (7.26), we have $\sum_i v_i(x_i) - \sum_i v(x_i) = -2 \sum_i \sum_{k \neq i} p_{ik} = 0$ and $\sum_{i < j} \{u_{ij}^+(x_i + x_j) + u_{ij}^-(x_i - x_j)\} = \sum_{i < j} \{u(x_i + x_j) + u(x_i - x_j)\}$. Hence $R(x)$ can be expressed as in (7.24).

Finally, because $u(t)$ is an even function, (7.13) implies that $v(t)$ is also an even function. \square

Remark 7.8 Applying (7.25) and the relation $v_i(t) = v(t) - 2 \sum_{k \neq i} p_{ik}$ to (7.22) and (7.23), we obtain

$$\begin{aligned} a_2^i &= - \sum_{\substack{j < k \\ \neq i}} \{u(x_j + x_k) + u(x_j - x_k)\} - \sum_{j \neq i} v(x_j), \\ a_{11}^{ij} &= -u(x_i + x_j) + u(x_i - x_j). \end{aligned}$$

Therefore, the functional equation (7.11) is identical to that studied by Ochiai, Oshima and Sekiguchi in [7, 10]. This equation has been completely solved, and the solutions are given in Theorem 1 of [8].

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